Random Walks

on Dynamical Percolation

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☕► All diagrams in this essay are snapshots of simulations coded in Python. I would like to thank dearly my essay setter Professor Sousi for her invaluable advice helping me navigate recent mathematical literature, as well as my best friend Sam Tomlin, for helping me with the simulations and putting up with me for so long.

1 Introduction

The random walk on a graph is a fundamental model in Probability Theory, not only because of its theoretical interest, but also due to its wide usage in the applied setting, such as estimating the size of the internet in computer science, to modelling polymers in physics. However, in practice, the phenomenon that the random walk attempts to capture, often doen't occur in an ideal, static medium, but rather in an ever changing environment whose structure influences the behavior of the walk. From the viewpoint of Probability, one way we may model this, is by adding randomness to the graph where the random walk is taking place. Amongst many possible choices for capturing the irregularities of the graph, Percolation - and in particular its dynamic version, introduced in [8] - is perhaps the most natural one.

This particular model, the *Random Walk on Dynamical Percolation* was introduced by Peres, Stauffer, and Steif in [16], and since its introduction the model has seen extensive investigation, mainly in the study of its mixing times (see for example [16], [15], [7]), as well as its hitting and cover times (see [16], [14], [7]) in the different regimes for percolation. However, an equally important property of the model is its diffusivity. In [16, Theorem 3.1], the authors proved a Central Limit Theorem for the random walk, showing that in the limit, the rescaled walk is normally distributed, from which one can show that the asymptotic growth of the mean square displacement of the walk is linear. In their paper, this diffusivity constant was studied in the subcritical regime, but more recently, in [5], significant progress has been made in understanding the diffusivity of the model in the other two regimes. With this as motivation, the goal of this essay will be to provide a concise overview of the model, studying in detail the diffusion of the walk.

The structure of this essay is the following: in Chapter 2 we will quickly introduce the model of dynamical percolation, including one result whose proof inspires some ideas which become useful later. In Chapter 3 we incorporate the random walk into the model, and prove some preliminary facts about the model. In particular, we will see that despite the addition of the randomly evolving environment, properties such as the dichotomy between transience and recurrence of the walk on $\mathbf{Z}^d$ still hold just as in the classical case. The proof of these two facts will lead us to introduce the important tool of regeneration times, culminating the chapter with a proof of the aforementioned Central Limit Theorem.

In Chapter 4 we will relate the Central Limit Theorem, and in particular, the variance of the limiting distribution, to the asymptotic growth rate of the mean square displacement: the diffusion constant. Just as the diffusion constant is related to the Central Limit Theorem, we will also discuss the notion of the speed, which plays a role analogous to that of a Strong Law of Large Numbers. The proofs involving the speed are often simpler than those for the diffusion constant despite containing most of the ideas. It is for this reason that we will often use it to illustrate our discussions.

Finally, in Chapters 5 and 6 we will discuss in detail the proofs for the known bounds for the diffusion constant and speed in the subcritical and critical cases respectively. Although bounds in the supercritical regime have also been achieved (see [5], and [6]), for reasons of brevity we will only present the results informally in Chapter 7, but not give their proof, since a discussion as detailed as the one we present in Chapters 5 and 6 would exceed the constraints of this essay.

2 Dynamical Percolation

Before introducing any random walks, we will briefly discuss the environment in which the walk will take place. This model for the environment, dynamical percolation, was first formalised by Häggström, Peres, and Steif [8], and one may think of it as the natural choice for a continuous-time Markov chain whose stationary distribution is that of the usual percolation model.

Definition 2.1 (Dynamical Percolation)**.** *The dynamical percolation model on a graph $G = (V, E)$ with parameters μ and p is a continuous time Markov Chain $(\eta_t)_{t\geq 0}$ on $\{0,1\}^E$, where each edge $e \in E$ evolves independently according to the following two-state Markov Chain: at the arrival times of a Poisson process with mean μ, set*

$$\eta_t(e) = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p. \end{cases}$$

We can also interpret this Markov chain by having each edge, independently from each other, evolve according to the two-state continuous-time with the following Q-matrix:

$$Q = \begin{pmatrix} -\mu(1-p) & \mu(1-p) \\ \mu p & -\mu p \end{pmatrix}$$

It is clear that the invariant distribution of the dynamical percolation chain will be the product measure $\bigotimes_{e\in E} \mathrm{Ber}(p)$, which we denote by π_p. Of course, the goal of this essay is not to explore the model of dynamical percolation itself, but rather how it affects the behavior of a random walk. Nonetheless, we now present a result [8, Proposition 1.1] that was simply too interesting to omit, since not only does it show how dynamical percolation shares similar properties with the static percolation model, but its proof employs technique that will be invaluable later:

Important Idea ☕. *In many of the scenarios we study in this essay, it is often useful to prove a result by first observing the evolution of the system during a small time interval, so that one can have a good control over the quantities of interest, and then extending the result to all values of time.*

Proposition 2.2. *Let $C \subset \{0,1\}^E$ be the event that there is an infinite cluster of open edges. Let $(\eta_t)_t$ be dynamical percolation on G, with parameters μ and p. Suppose that $\eta_0 \sim \pi_p$. Let p_c denote the value of critical bond percolation on the graph G, then*

$$\begin{cases} \mathbf{P}[\eta_t \in C \text{ for all } t \geq 0] = 1 & p > p_c \\ \mathbf{P}[\eta_t \in C^c \text{ for all } t \geq 0] = 1 & p < p_c \end{cases}$$

Proof. For simplicity we assume that the refresh rate μ is equal to one. The ingenuous idea is to couple our dynamical percolation $(\eta_t)_t$ with a static percolation formed by all edges that have been constantly open during a small time interval, and then use facts from static percolation to finish the proof. We only show the claim for the supercritical case, since the subcritical case follows with a very similar argument. Choose $\epsilon > 0$ small enough so that $p - \epsilon$ is still strictly larger than p_c, and define $\eta^{[0,\epsilon]}$ to be static percolation on G where we declare an edge $e \in E$ to be open, i.e: $\eta^{[0,\epsilon]}(e) = 1$ if $\eta_t(e) = 1$ for all $t \in [0, \epsilon]$, so that in effect, this is static percolation with parameter

$$\begin{aligned}\mathbf{P}\left[\inf_{t\in[0,\epsilon]}\eta_t(e)=1\right]&=\mathbf{P}\left[\{\eta_0(e)=1\}\cap\{\eta_t(e)\text{ does not transition to }0\text{ for all }t\in[0,\epsilon]\}\right]\\&=p\exp(-(1-p)\epsilon)\\&>p-\epsilon>p_c\end{aligned}$$

The way the coupling has been constructed means that for all $t\in[0,\epsilon]$, $\eta_t\geq\eta^{[0,\epsilon]}$, so in particular, if $\eta^{[0,\epsilon]}\in C$, then $\eta_t\in C$ for all $t\in[0,\epsilon]$. Since $\eta^{[0,\epsilon]}$ is supercritical percolation, we automatically get that $\mathbf{P}[\eta_t\in C\text{ for all }t\in[0,\epsilon]]=1$. Now the next trick is to extend this result to all values of t. This comes from a simple sub-additivity argument:

$$\begin{aligned}\mathbf{P}[\eta_t\in C\text{ for all }t\geq 0]&=\mathbf{P}\left[\bigcap_{k\geq 0}\{\eta_t\in C\text{ for all }t\in[k\epsilon,(k+1)\epsilon]\}\right]\\&\geq 1-\sum_{k\geq 0}\mathbf{P}[\{\eta_t\in C\text{ for all }t\in[k\epsilon,(k+1)\epsilon]\}^c]=1\end{aligned}$$

Where effectively we have used that one can repeat the argument above to work in any interval of the form $[k\epsilon,(k+1)\epsilon]$. □

Remark 2.3. *We will not delve further into questions of dynamical percolation alone, but as some concluding remarks, the natural question to ask now would be, what happens at p_c? Loosely speaking, the authors of [8] focus on this case by studying, at criticality, the probability*

$$\mathbf{P}\left[\bigcup_{t>0}\{\eta_t\in C\}\right]$$

the authors very appropriately use the term "flickering" to denote a graph for which this probability is one. In their own words: "for some exceptional times, the infinite cluster flickers by". In [8, Theorem 1.2], they show the existence of a flickering graph.

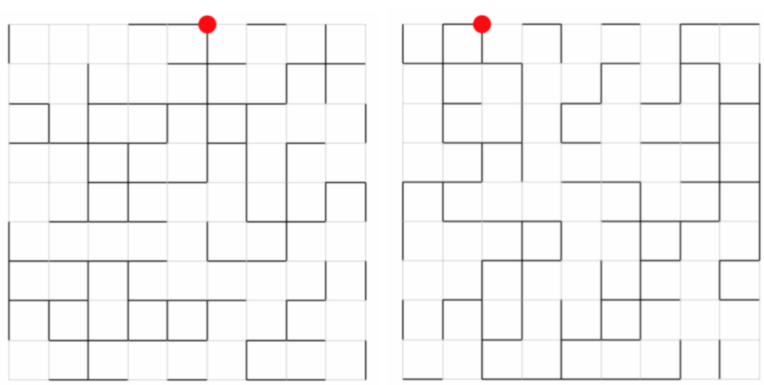

Figure 1: Two snapshots of a simulation of a random walk on dynamical percolation on a box.

3 The Random Walk: Transience, Recurrence, and a CLT

Definition 3.1 (Random Walk on Dynamical Percolation). *Let $G = (V, E)$ be a bounded degree graph, and let $(\eta_t)_t$ be a dynamical percolation on G. The Random Walk on Dynamical Percolation model is a process $(X_t)_t$ on V which at arrival times of a Poisson process with mean 1, attempts a jump by choosing uniformly at random one of its neighbouring edges. If the chosen edge is open at the time of the jump, the particle changes location; otherwise it stays in the same position.*

Unless otherwise specified, $(X_t)_{t\geq 0}$ will always denote the random walk on dynamical percolation. We may sometimes refer to the random walk as a *particle*.

Remark 3.2 (A Markov chain?). *An important detail is that while the joint process $(X_t, \eta_t)_t$ is a continuous time Markov chain on $V \times \{0,1\}^E$, the walk itself $(X_t)_t$ is not a Markov chain on V because it has a "memory". Indeed, the jumps that the particle has been able to make give some information on the current bond configuration of the graph, thus influencing the future distribution of X.*

Transience and Recurrence

The goal of this section is to establish some preliminary results about the walker $(X_t)_{t\geq 0}$. We will start by showcasing how despite the newly added randomly evolving environment, we are still able to deduce familiar results. The starting point for us will be to show that the transience/recurrence dichotomy [16, Theorem 1.1] still holds. In a very similar spirit to Proposition 2.2, we don't include this result because its crucial for the remaining of this essay, but rather, we include this result because its proof introduces an idea which we will use later on to establish that the walk exhibits diffusive behavior:

Important Idea ☕. *Classic random walks have been studied extensively, and we have plenty of tools to analyse them. However, in light of Remark 3.2, we can't immediately apply these tools due to the "memory" of the walk. The ingenious step in the proof of Theorem 3.4 is to construct a sequence of stopping times, at which the walk no longer knows the state of the bonds which it has seen. This intuitively tells us that the walk observed at these times behaves like a random walk (or better said, a random flight). We will then be able to use classic tools along these times, and then the rest of the work will go into showing that one can "safely interpolate" between the usual time and these special times.*

Let us briefly recall a classic result [10, Theorem 4.1.1] about random walks on $\mathbf{Z}^d$. Note that whenever we say "random walks on $\mathbf{Z}^d$", we refer to a process taking values on $\mathbf{Z}^d$ whose increments are independent and identically distributed.

Lemma 3.3. *If $(X_n)_n$ is a discrete time irreducible aperiodic random walk on $\mathbf{Z}^d$, then:*

- *If $d \in \{1, 2\}$, and the increments of the random walk have zero mean and finite second moment, then the walk is recurrent, i.e:*

$$\mathbf{P}[\{X_n = 0 \text{ infinitely often}\}] = 1.$$

- *If $d \geq 3$, then the walk is transient.*

We now state the result for Dynamical Percolation.

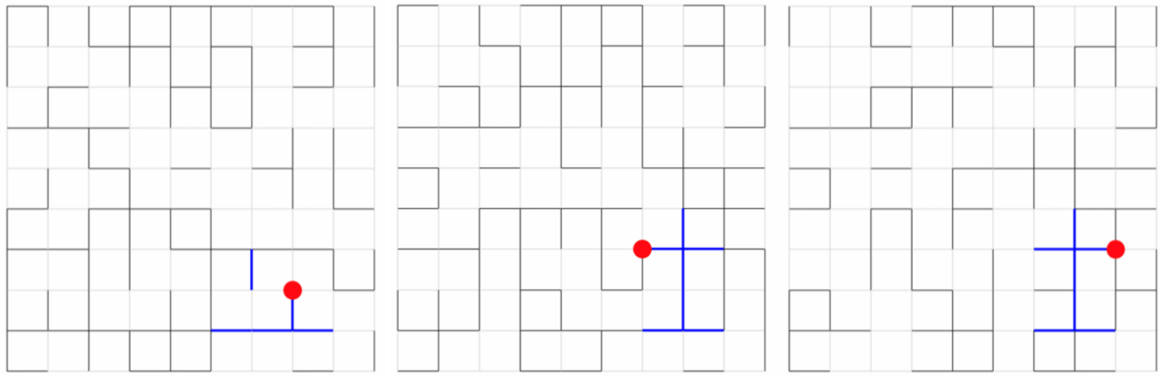

Figure 2: Three snapshots of a random walk on dynamical percolation where the edges in the memory set at each time are highlighted in blue

Theorem 3.4 (Transience and Recurrence)**.** *Let $(X_t)_t$ be a random walk on dynamical percolation in $\mathbf{Z}^d$, with parameters $p \in (0,1]$ and $\mu > 0$. Then for any initial bond configuration $\eta_0 \in \{0,1\}^{E(\mathbf{Z}^d)}$, we have the following:*

- *(Recurrence) If $d \in \{1,2\}$, then for any $s_0 > 0$, we have that*

$$\mathbf{P}\left[\bigcup_{s>s_0} \{X_s = 0\}\right] = 1.$$

- *(Transience) If $d \geq 3$, then we have that almost surely*

$$\lim_{n\to\infty} \|X_n\| = \infty.$$

Before we begin the proof, we introduce some useful notation:

- For an edge $e \in E$, denote by $0 \leq \chi_1^e < \chi_2^e < \cdots$ the sequence of refresh times of the edge.
- Let $0 \leq \xi_1 < \xi_2 < \cdots$ denote the sequence of times when the random walk on dynamical percolation $(X_t)_t$ attempted to perform a jump. we also let $\mathbf{e}_k$ to denote the k^{th} edge that the walker attempted to jump.

Proof of recurrence. For simplicity, we will prove the claim for initial bond configuration η_0 distributed according to π_p, which in effect means that the bond distribution is π_p at all times. Nonetheless this can be extended (see the end of the proof of [16, Theorem 1.1]) to arbitrary initial bond configuration. We also assume for convenience that $X_0 = 0$. Consider a process $(A_t)_{t\geq 0}$ of "memory sets", which intuitively speaking, store the edges for which the random walk $(X_t)_t$ has information about at a given time. Formally: first let $A_0 = \varnothing$, then let

$$A_t = \left\{ e \in E(\mathbf{Z}^d) : \max_k \left\{ \chi_j^e : \chi_j^e \leq t \right\} < \max_k \left\{ \xi_k : \xi_k \leq t, \mathbf{e}_k = e \right\} \right\},$$

that is to say, an edge e belongs to A_t if its last refresh time before time t occurred before the last jump attempt of the particle to e during this same time interval, so that in effect, the particle *remembers* the state of the edge at time t. As mentioned above, the key idea of the proof, is that at the times when the memory set is empty, the walk X behaves like random walk in discrete time. These times are called *regeneration times*, and we can formalise this idea as follows: let $\tau_0 = 0$ and define inductively

$$\tau_k = \inf\left\{t > \tau_{k-1} : |A_t| = 0, \sup_{s\in(\tau_{k-1},t)} |A_s| \geq 1\right\},$$

that is to say, the times at which the memory set becomes empty after having been nonempty for some time. These are stopping times with respect to the filtration

$$\mathcal{F}_t = \sigma\left((X_s, \eta_s)_{0\leq s\leq t}, (\chi_k^e)_{k\in\mathbf{N}}, (\xi_j, \mathbf{e}_j)_{j\in\mathbf{N}}, e \in E\right),$$

and it is clear, by the fact that we started the bond distribution at stationarity and the fact that $\mathbf{Z}^d$ is a transitive graph, that the sequence $(\tau_{k+1} - \tau_k)_{k\geq 0}$ is an i.i.d sequence. It is now easy to verify, using the Strong Markov Property on the joint process $(X_t, \eta_t)_t$, that the increments $(U_k)_{k\geq 0} := (X_{\tau_k} - X_{\tau_{k-1}})_{k\geq 0}$ form an i.i.d sequence with distribution $U_1 := X_{\tau_1}$. This means that the walk observed at the regeneration times, $(X_{\tau_k})_{k\geq 0}$, is in fact a discrete-time random walk. As mentioned before, this will not be necessarily a simple symmetric random walk, it will typically be the case that the walk performs long jumps when observed at the regeneration times. We now move on to check that the conditions of Lemma 3.3 are satisfied and conclude the proof, for if $(X_{\tau_k})_{k\geq 0}$ is recurrent, then so is $(X_t)_{t\geq 0}$. First of all, it is clear that U_1 takes the value 0 as well as any of the $2d$ neighbours of zero with positive probability, which means that the random walk is irreducible and aperiodic. We now sketch how to verify the condition on the second moments: if we define $Z_k := \#\{$attempted jumps in the time interval $[\tau_{k-1}, \tau_k]\}$, then it is clear that $\text{dist}(U_1, 0) \leq Z_1$. By a simple birth and death chain argument, see for example [7, Lemma 3.5], it can be shown that $(\tau_k - \tau_{k-1})_{k\geq 1}$ have exponential tails, so from the exponential concentration of a Poisson random variable around its mean, it follows that Z_1 has exponential tails too, and so $\mathbf{E}[\text{dist}(U_1), 0)^2] < \infty$ as required.

□

We now move on to study the behavior of this walk in the case $d \geq 3$, for this, we will need another background result [10, 2.1.1], which we slightly restate for the purposes of our proof:

Theorem 3.5 (Local Central Limit Theorem). *Let $(X_n)_{n\geq 0}$ be a (discrete time) irreducible, aperiodic, centered random walk on $\mathbf{Z}^d$ whose increments have finite second moments, then for $x \in \mathbf{Z}^d$ with dist$(x, 0) \leq \sqrt{n}$, one has that*

$$\mathbf{P}[X_n = x] \asymp n^{-d/2}$$

Proof of transience. Using the regeneration times $(\tau_k)_{k\geq 0}$ as before, it follows automatically from Lemma 3.3 that the random walk $(X_{\tau_k})_{k\geq 0}$ is transient for $d \geq 3$. However, it could still be the case that in between the times τ_k and τ_{k+1}, the walk returned all the way to zero. For this, fix some radius $R > 0$ and consider $\mathbf{B}(0, R)$ to be the ball around zero (in graph distance). We want to consider the events

$$E_k = \{X_t \in \mathbf{B}(0, R) \text{ for some } t \in [\tau_k, \tau_{k+1}]\}$$

and show that they cannot hold infinitely often. This will be a simple consequence of Borel-Cantelli and the Local Central Limit Theorem stated above. First of all, it is easy to see that we can write

$$\sum_{k\geq 0} \mathbf{P}[E_k] \leq \sum_{k\geq 0} \mathbf{P}\left[E_k \middle| \text{dist}(X_{\tau_k,0)} \geq k^{1/4d}\right] + \mathbf{P}\left[\text{dist}(X_{\tau_k}, 0) \leq k^{1/4d}\right] \tag{3.1}$$

Let us analyse the first term in the sum: if the walk X_{τ_k} is currently further than $k^{1/4d}$ from the origin and before time τ_{k+1} it returns to the ball of radius R, then the walk on dynamical percolation must have attempted at least $k^{1/4d} - R$ jumps (recall we used Z_k to denote the number of attempted jumps in between two successive regeneration times) . From this we get that

$$\mathbf{P}\left[E_k \middle| \text{dist}(X_{\tau_k,0}) \geq k^{1/4d}\right] \leq \mathbf{P}\left[Z_1 \geq k^{1/4d} - R\right]$$

but now we can apply a Chernoff bound along the fact that as we mentioned, it can be shown that $\mathbf{E}[\exp(cZ_1)] < \infty$ for some c, in order to show that this probability decays exponentially, so the first term in the sum of equation 3.1 is summable. To take care of the second term, we simply note that

$$\mathbf{P}\left[\text{dist}(X_{\tau_k}, 0) \leq k^{1/4d}\right] = \sum_{x \in \mathbf{B}(0, k^{1/4d})} \mathbf{P}[X_{\tau_k} = x] \tag{3.2}$$

now, since all the values of x that we are summing over are at a distance less than $\sqrt{k}$, the Local Central Limit Theorem applies (recall that we already "checked" all other conditions in the proof of recurrence), and since the number of points inside $\mathbf{B}(0, k^{1/4d})$ grows like $k^{1/4}$, we have that the probability in 3.2 decays as fast as $\frac{k^{1/4}}{k^{d/2}}$ and since $d \geq 3$ by assumption, we have that 3.2 is in fact also summable, and so by Borel-Cantelli we get that

$$\mathbf{P}[E_k \text{ infinitely often}] = 0$$

countable additivity now finishes the claim. □

A Central Limit Theorem

The Central Limit Theorem states that if $X_1, X_2 \cdots$, are i.i.d centered real random variables with variance σ^2, then the distribution of

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

becomes the distribution of a $\mathcal{N}(0, \sigma^2)$ random variable as $n \to \infty$. One can now ask whether our scenario of random walks on dynamical percolation admits an analogous result. Naturally, the challenge will be to overcome the correlations due to the environment, so in a similar fashion as in the proof of Theorem 3.4, we will exploit the regeneration times $(\tau_k)_{k \geq 0}$, along which we will have the desired convergence. The rest of the proof will follow by once again showing that one can in some sense interpolate between any given time k and a suitable regeneration time. In [16, Theorem 3.1], the result is stated as a convergence of the rescaled walk to a Brownian motion in distribution, but for our purposes something weaker will suffice. In the original paper, the proof of this Theorem is presented as a sketch, so to present a full proof we use slightly different argument inspired by the techniques employed in [1, Proposition 3.2]

Theorem 3.6 (Central Limit Theorem for dynamical percolation). *Let $(X_t)_{t \geq 0}$ be a random walk on dynamical percolation on $\mathbf{Z}^d$ with parameters p and μ. Then there exists some constant $\sigma = \sigma(d, p, \mu) \in (0, \infty)$ such that*

$$\left(\frac{X_k}{\sqrt{k}}\right) \xrightarrow{(d)} \mathcal{N}(0, \sigma^2 \mathbf{I})$$

where $\mathbf{I}$ is the $d \times d$ identity matrix.

Proof. Recall that as we discussed in the proof of Theorem 3.4, $(X_{\tau_k})_{k \geq 0}$ is indeed a random walk given by $X_{\tau_n} = \sum_{i=1}^{n} U_i$, where $(U_i)_{i \geq 0}$ were the i.i.d, zero-mean, step increments of the walk. First and foremost we recall the Random Index Central Limit Theorem [3, Exercise 3.4.6], that in our case implies that if $\ell(k)$ is a sequence of non-negative, integer-valued random variables and a_k is a sequence of integers with $a_k \to \infty$ and $\ell(k)/a_k \to 1$ in probability, then:

$$\frac{X_{\tau_{l(k)}}}{\sqrt{a_k}} \overset{(d)}{\to} \mathcal{N}\left(0, \mathsf{Var}(U_1^{(1)})\mathbf{I}\right).$$

We may now define $\ell(k) = \max\{l \in \mathbf{N} : \tau_l \leq k\}$, i.e: the last regeneration time before time k, and rewrite:

$$\frac{X_k}{\sqrt{k}} = \frac{X_k - X_{\tau_{\ell(k)}}}{\sqrt{k}} - \frac{X_{\tau_{\ell(k)}}}{\sqrt{k}} \frac{\sqrt{\mathbf{E}[\tau_1]}}{\sqrt{\mathbf{E}[\tau_1]}}. \tag{3.3}$$

By writing $J[s,t] \sim \mathsf{Poi}(t-s)$ to be the number of jump attempts the random walk on dynamical percolation has made between times s and t, we have that $|X_k - X_{\tau_{\ell(k)}}| \leq J[\tau_{\ell(k)}, \tau_{\ell(k)+1}]$, which in turn has the same distribution as the random variable Z_1 which we defined in the proof of Theorem 3.4 to be the number of jump attempts between time zero and the first regeneration time. We saw how Z_1 had finite variance, which implies, say by Markov's Inequality that the first term of 3.3 converges to zero in probability, and hence in distribution. For the second term of 3.3, we note that it is an easy computation to verify that $\ell(k)/k \to 1/\mathbf{E}[\tau_1]$ almost surely, which by the Random Index Central Limit Theorem, gives us that the second term of 3.3 converges to a normal random variable of variance $\frac{\mathsf{Var}(U_1^{(1)})}{\mathbf{E}[\tau_1]}$. Then we can combine these two convergences with Slutsky's Theorem and get that $X_k/\sqrt{k}$ has the correct convergence. $\square$

4 The diffusion constant and the speed of the walk

Random walk on dynamical percolation is diffusive

The convergence in distribution to a normal random variable with variance σ^2 proved in Theorem 3.6 hints to the fact that the random walk on dynamical percolation also exhibits diffusive behavior. By that we mean that $\mathbf{E}[\text{dist}(0, X_t)^2] \sim Ct$ for some constant C as $t \to \infty$. As shown in the following proposition, this is indeed the case, and as we might expect, the **diffusion constant** C corresponds precisely to the variance σ^2 from Theorem 3.6. This fact will leads us naturally to ask how σ^2 behaves for the different parameters of the model, which will be the central topic of discussion for the remaining of the essay.

Proposition 4.1. *Let $(X_t)_{t\geq 0}$ be a random walk on dynamical percolation on $\mathbf{Z}^d$. Let σ^2 be the diffusion constant of Theorem 3.6, then*

$$\sigma^2(d,p) = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}[d(0,X_t)^2]$$

where d is the Euclidean distance.

Remark 4.2. *In the above proposition, we use the Euclidean distance to establish the limit, but throughout the remaining of this essay, we will use the graph distance on $\mathbf{Z}^d$. This is not an inconvenience, since we will only be interested in the behavior of σ^2 up to constants, and the Euclidean distance and the graph distance on $\mathbf{Z}^d$ are equivalent metrics.*

One of the final steps in the following argument relies on a result that will be proven later in Section 5. However, since Proposition 4.1 is mostly to illustrate the connection between Theorem 3.6 and the mean square displacement, we present the proof now to preserve the flow of the essay. To the best of the writer's knowledge, a proof of this fact has not been given in the literature, although in [5], some hints are given as to how this can be shown. The proof we give here, (inspired once again by the techniques of the proof of [1, Proposition 3.2]) relies on observing the walk at the regeneration times, and then controlling the distance between a given time and an appropriate regeneration time.

Proof of Proposition 4.1. Let $(\tau_k)_{k\in\mathbf{N}}$ be the sequence of regeneration times for the walk and write, as in the proof of Theorem 3.6, $\ell(t)$ to be the index of the last regeneration time before time t. We can write

$$\frac{\mathbf{E}[d(0,X_t)^2]}{t} = \frac{\mathbf{E}[d(0,X_t)^2 - d(0,X_{\tau_{\ell(t)}})^2]}{t} + \frac{\mathbf{E}[d(0,X_{\tau_{\ell(t)}})^2]}{t}. \tag{4.1}$$

Now observe that since $(X_{\tau_k})_k$ is a random walk with i.i.d increments, then by writing $X_{\tau_{\ell(t)}}$ as a sum of the increments and using Wald's identity, we have that $\mathbf{E}[d(0,X_{\tau_{\ell(t)}})^2] = \mathbf{E}[\ell(t)]\text{Var}(U_1)$. By a standard fact in renewal theory, we also have that $\frac{\mathbf{E}[\ell(t)]}{t} \to \frac{1}{\mathbf{E}[\tau_1]}$, which gives that the second term of the right hand side in 4.1 does indeed converge to σ^2. To take care of the first term in the right hand side, we need to control the differences of the squares of the distances between time t and time $\tau_{\ell(t)}$. For this we have the following computation:

$$\begin{aligned}
|\mathbf{E}[d(0,X_t)^2 - d(0,X_{\tau_{\ell(t)}})^2]| &= \left|\mathbf{E}\left[\left(d(0,X_t) - d(0,X_{\tau_{\ell(t)}})\right)\left(d(0,X_t) + d(0,X_{\tau_{\ell(t)}})\right)\right]\right| && (4.2)\\
&\leq \sqrt{\mathbf{E}\left[\left(d(0,X_t) - d(0,X_{\tau_{\ell(t)}})\right)^2\right]\mathbf{E}\left[\left(d(0,X_t) + d(0,X_{\tau_{\ell(t)}})\right)^2\right]} && (4.3)\\
&\leq 2\sqrt{\mathbf{E}\left[d(X_t,X_{\tau_{\ell(t)}})^2\right]\mathbf{E}\left[d(0,X_t)^2 + d(0,X_{\tau_{\ell(t)}})^2\right]} && (4.4)
\end{aligned}$$

Where step 4.3 is the Cauchy-Schwarz inequality, and step 4.4 used the triangle inequality as well as the fact that for real numbers a, b one always has that $(a+b)^2 \leq 4(a^2+b^2)$. Now observe that the first expectation of 4.4 is bounded above by $\mathbf{E}[J(\tau_{\ell(t)}, \tau_{\ell(t)+1})^2]$ which is just the variance of the number of jumps between two consecutive regeneration times, and we know this to be finite as mentioned in the proof of Theorem 3.6. As mentioned just before this proof started, now we will have to make a forward-reference. As we will show in Proposition 5.9, for a random walk on dynamical percolation we have the general upper bound which tells us that $\mathbf{E}[d(0,X_t)^2]$ is $O(t)$. Moreover, as we have also seen in this proof, $\mathbf{E}[d(0, X_{\tau_{\ell(k)}})^2] \lesssim \mathbf{E}[\ell(t)]$, which is also $O(t)$. Therefore, due to the square root, we see that the whole term in 4.4 is $o(t)$. Combining this back into 4.1 and taking $t \to \infty$ gives the claim. $\square$

The speed of the random walk

As explained above, the rate of growth of the mean square displacement is an object of interest to understand the asymptotic properties of the motion of our walk. Another interesting question is whether the displacement itself scales linearly in the limit as $t \to \infty$. As we will see in the next result [6, Lemma 2.3], this is indeed true, and the resulting ratio in the limit is called the **speed** of the random walk. As mentioned in the introduction to this essay, we don't just include this adjacent discussion for its own sake despite its theoretical interest; indeed, as we will see specially in Chapter 6, proofs involving the speed involve often simpler computations than those for the diffusion constant, despite containing essentially the same conclusions and ideas. Therefore, despite the central topic of the essay being the diffusion constant, we will illustrate proofs in context of the speed whenever it aids comprehension. Let us now state and prove the following Theorem:

Theorem 4.3 (Speed of the random walk)**.** *Let $(X_t)_{t\geq 0}$ be a random walk on dynamical percolation on an infinite transitive graph G with initial bond configuration distributed according to π_p. Then there is a constant $v = v(p, \mu, d)$ such that*

$$\lim_{t\to\infty} \frac{\textit{dist}(0, X_t)}{t} = v. \tag{4.5}$$

Where the limit is almost-sure and in $\mathcal{L}^1$.

Proof. The idea for the proof of this limit is to use the celebrated subadditive Ergodic Theorem, which is commonly used in such proofs where one deals with the scaling behavior of distances. The subadditive Ergodic Theorem (see [12] for a reference) concerns a family of real valued random variables $(Z_{n,m})_{n,m\geq 0}$ that satisfies the subadditive property:

$$Z_{n,m} \leq Z_{n,k} + Z_{k,m}$$

for any $n+1 \leq k \leq m-1$, in addition to $\mathbf{E}[Z_{0,n}] \in [-Cn, \infty)$ for some $C \geq 0$ as well as $(Z_{n,m})_{n,m\geq 0}$ being stationary, that is to say: the distribution of $(X_{n+k,n+m+k})_{n,m\geq 0}$ is independent of $k \in \mathbf{N}$. If these conditions are satisfied, then the subadditive Ergodic Theorem tells us that

$$\lim_{n\to\infty} \frac{X_{0,n}}{n}$$

converges almost surely. In addition to this, if any event defined in terms of our family of random variables $(X_{n,m})_{n,m\geq 0}$ that is invariant under the shift $(n,m) \mapsto (n+1, m+1)$ has probability zero or one, then it follows that the limit is also constant almost surely. The way we will prove 4.5 is by establishing the limit first along the regeneration times $(\tau_n)_{n\geq 0}$, which for the reader's convenience, we redefine here: recall that the memory set at time t, A_t, is defined as

$$A_t = \left\{ e \in E(\mathbf{Z}^d) : \max_k \left\{ \chi_j^e : \chi_j^e \leq t \right\} < \max_k \left\{ \xi_k : \xi_k \leq t, \mathbf{e}_k = e \right\} \right\},$$

where χ_i^e was the i^{th} refresh time of an edge $e \in E$, ξ_i was the time of the i^{th} jump attempt of the particle, and $\mathbf{e}_i$ was the i^{th} attempted edge. Then the regeneration times $(\tau_k)_{k \in \mathbf{N}}$ were inductively defined by

$$\tau_k = \inf\left\{ t > \tau_{k-1} : |A_t| = 0, \sup_{s \in (\tau_{k-1}, t)} |A_s| \geq 1 \right\},$$

and $\tau_0 = 0$. To establish the limit along this sequence we will use the subadditive Ergodic Theorem with the family of random variables $(\text{dist}(X_{\tau_n}, X_{\tau_m}))_{n,m \geq 0}$. Subadditivity, as well as the ergodicity properties are immediate from the triangle inequality, and the fact that $(X_{\tau_n})_{n \geq 0}$ is a random walk on G. Therefore the only remaining fact to check is the integrability condition. Since distances take non-negative values, all we are interested in showing is that $\mathbf{E}[\text{dist}(0, X_{\tau_n})] < \infty$ for all n. This will follow from the next observations:

1. Recall that $(\tau_n)_{n \geq 0}$ have the property that $(\tau_i - \tau_{i-1})_{i>1}$ are i.i.d, and so $\mathbf{E}[\tau_n] = n\mathbf{E}[\tau_1]$.

2. As briefly hinted in the proof of Theorem 3.4, the memory set process A_t is stochastically dominated by a birth-death process S_t with birth rate 1 and death rate $\mu|S_t|$. Hence if we define $\tilde{\tau}_1$ to be the first "extinction time" of the process S_t, we have that $\mathbf{E}[\tau_1] \leq \mathbf{E}[\tilde{\tau}_1]$. Moreover, by a simple calculation (see [6, Page 8]), one can show that $\mathbf{E}[\tilde{\tau}_1] \leq \exp(1/\mu)$.

3. The increments of the process $(\text{dist}(0, X_t))_{t \geq 0}$ have rate at most 1, from which it easily follows that $(\text{dist}(0, X_t) - t)_{t \geq 0}$ is a supermartingale. Then we have that

$$\begin{aligned} \mathbf{E}[\text{dist}(0, X_{\tau_n})] &\leq \liminf_{t \to \infty} \mathbf{E}[\text{dist}(0, X_{\tau_n \wedge t})] && (4.6) \\ &\leq \liminf_{t \to \infty} \mathbf{E}[\tau_n \wedge t] && (4.7) \\ &\leq \mathbf{E}[\tau_n] && (4.8) \end{aligned}$$

   Where 4.6 is Fatou's Lemma, 4.7 is the Optional Stopping Theorem, and 4.8 is the Monotone Convergence Theorem.

Combining these three observations, the integrability condition on $\text{dist}(0, X_{\tau_n})$ we were seeking is satisfied, and so the subadditive Ergodic Theorem finishes the claim for the limit along $(\tau_n)_{n \geq 0}$, call this limit v'. To extend the limit to all values of $\mathbf{R}^+$ we just need to control the value of the limit between reset times. For this we make the following simple observations:

1. If we let $J[s, t]$ be the number of attempted jumps in the interval $[s, t]$, we then have that

$$\begin{aligned} \frac{\text{dist}(0, X_{\tau_n}) - J[\tau_{n-1}, \tau_n]}{\tau_n} &\leq \inf_{s \in (\tau_{n-1}, \tau_n]} \frac{\text{dist}(0, X_s)}{s} \\ &\leq \sup_{s \in (\tau_{n-1}, \tau_n]} \frac{\text{dist}(0, X_s)}{s} \leq \frac{\text{dist}(0, X_{\tau_{n-1}}) - J[\tau_{n-1}, \tau_n]}{\tau_{n-1}} \end{aligned} \qquad (4.9)$$

2. The process $J_t = J[0, t]$ is a Poisson point process on $\mathbf{R}+$ with intensity 1, so by [3, Theorem 2.4.7], we have the following Law of Large Numbers:

$$\lim_{t \to \infty} \frac{J[0, t]}{t} = 1 \quad \text{almost surely.}$$

3. Using again the fact that $(\tau_i - \tau_{i-1})_{i \in \mathbf{N}}$ are i.i.d, we have that by the Law of Large Numbers, both τ_n/τ_{n-1} and τ_{n-1}/τ_n converge to 1 almost surely. Therefore we have the almost sure limit

$$\frac{J[\tau_{n-1}, \tau_n]}{\tau_n} = \frac{J[0, \tau_n]}{\tau_n} - \frac{J[0, \tau_{n-1}]}{\tau_{n-1}} \frac{\tau_{n-1}}{\tau_n} \to 0,$$

   and similarly if τ_{n-1} had been in the denominator of the first fraction.

4. Finally, once again, using the law of the large numbers, we have the almost sure limit:

$$\frac{\text{dist}(0, X_{\tau_n})}{\tau_n} = \frac{\text{dist}(0, X_{\tau_n})}{n} \frac{n}{\tau_n} \to \frac{v'}{\mathbf{E}[\tau_1]}.$$

Putting these observations together, we show that the left-most and right-most terms of 4.9 are in fact the same, thus establishing the almost sure convergence along the real numbers. To upgrade this convergence to $\mathcal{L}^1$ convergence, we simply note that $\text{dist}(0, X_t)$ being dominated by a Poisson random variable with parameter t, easily implies that $\left(\frac{\text{dist}(0,X_t)}{t}\right)_t$ is $\mathcal{L}^2$ bounded and so Uniformly Integrable, this gives the desired $\mathcal{L}^1$ convergence. $\square$

Remark 4.4 (Initial bond configuration)**.** *For clarity, the Theorem above was presented in the concrete case of the bond configuration having stationary initial distribution. This was crucial for the regeneration times to turn $(X_t)_{t\geq 0}$ into a random walk. However, the authors of [6] treat the more general case, where they show that the initial bond configuration can be taken to be any $\eta_0 \in \{0,1\}^{E(G)}$. The fact that the claim holds for any initial environment is a consequence of [6, Proposition 2.1]*

Remark 4.5 (Speed on $\mathbf{Z}^d$)**.** *The Theorem above was stated for general graphs G that are infinite and transitive. We can quickly remark that the case of $\mathbf{Z}^d$ is not very interesting, since the speed is actually zero. Indeed: suppose it were not, let $\epsilon > 0$ be small enough so that $v - \epsilon > 0$. Then by Theorem 4.3 we have that*

$$\mathbf{P}\left[\frac{\textit{dist}(0, X_t)}{t} > v - \epsilon\right] \to 1,$$

but on the other hand,

$$\mathbf{P}\left[\frac{\textit{dist}(0, X_t)}{t} > v - \epsilon\right] \leq \frac{\mathbf{E}[\textit{dist}(0, X_t)^2]}{t^2(v-e)^2} \lesssim \frac{1}{t} \to 0.$$

5 Subcritical Regime

Having established the preliminary results concerning σ^2 and v, we proceed to our study of these quantities in the different regimes of p. The natural place to start, as is usually the case in percolation, is with the subcritical regime, since the exponential decay of the size of the clusters can facilitate our study of the motion of the random walk. The result concerning mean square displacement which we state now, was first proven in [16, Theorem 1.4], for the case where $G = \mathbf{Z}_n^d$. The result can then be extended to the whole of $\mathbf{Z}^d$.

Theorem 5.1 (Mean-square-displacement on subcritical regime)**.** *Fix d and let $p \in (0, p_c(\mathbf{Z}^d))$. Then for all n, μ, t, we have that for the random walk X_t on the dynamical percolation on $\mathbf{Z}_n^d$ started with $u \otimes \pi_p$ satisfies:*

$$\mathbf{E}\left[d\left(X_t, X_0\right)^2\right] \lesssim (\mu t) \vee 1$$

Naturally, we immediately get that

Corollary 5.2 (Diffusion constant on subcritical regime)**.** *The diffusion constant σ^2 of a random walk on dynamical percolation in $\mathbf{Z}_n^d$ started at stationarity with $p < p_c$ is $O(\mu)$.*

As for the speed, we have the following result [6, Theorem 1.1]:

Theorem 5.3 (Speed on subcritical regime)**.** *Let G be a connected, locally infinite, non-amenable, transitive and unimodular graph where each vertex has degree $d \geq 3$. We then have that the speed of the walk is order $\mu \wedge 1$.*

For reasons of brevity, in this essay we include only the proof of Theorem 5.1. The corresponding upper bound for the speed follows an almost identical argument, while the lower bound for the speed is more intricate and requires other tools such as the *evolving set process*.

The upper bound: heuristics

We now briefly comment on the main idea of the proof:

Important Idea ☕. *Since we work in subcritical percolation, we know that the clusters are small. We can use this to our advantage by considering a very special cluster: suppose that we let the system run for a very small amount of time $[0, s]$, and we let $\mathcal{H}$ be all the bonds that have been open at least once during this interval of time. If we choose s to be small enough, then this picture of bonds will still be that of subcritical percolation, and since the particle is constrained to move only in $\mathcal{H}$ (see Figure 3), we will deduce that the particle can't have moved very far away from its starting position.*

This philosophy of letting our system run for only a cleverly chosen period of time and should seem natural after recalling how we proved Theorem 2.2. The rest of the work will come when we want to extend this result, which we obtained only in a small time interval $[0, s]$, to also hold in any time interval. For this, we will need to explore the so-called Markov type property of metric spaces, a powerful tool introduced by Ball in [2], which we will develop in detail in due course. We now formalise this first step:

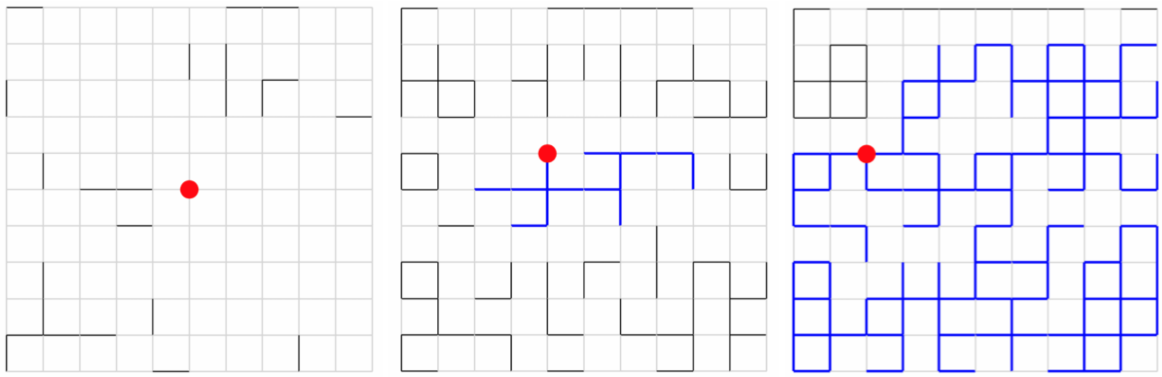

Figure 3: Simulation of the bonds that have been open at least once for increasing values of time: in red is the random walk and blue is the cluster of zero obtained from $\mathcal{H}$.

Proof of the upper bound

Step 1: Control of dist$(0, X_t)$ in a small time interval

As defined in the heuristics of the proof, we let the system run for a small amount of time, and then analyse what has happened in this interval. Observe that for a given edge $e \in E(\mathbf{Z}_n^d)$, the probability of e being open at least once in the time interval $[0,t]$ equals $1 - \mathbf{P}[e$ always closed in $[0,t]]$ and since we know that edges open at rate μp, we can easily obtain that

$$\mathbf{P}[e \text{ open at least once in } [0,t]] = 1 - (1-p)\left(\exp(-\mu t p)\right) \leq p(1+\mu t) \tag{5.1}$$

We now wish to choose a value of t small enough so that this probability is still subcritical, so if we choose β small enough so that $p(1+\beta) < p_c$, we see that the probability that an edge e has been open at some point in the time interval $[0, \beta/\mu]$ is both subcritical, and independent of μ (and obviously of n). Recall that now $\mathcal{H}$ will be used to denote the edges that have been open at least once in $[0, \beta/\mu]$. Here we have a subtlety, namely that the interval we have chosen depends on μ. To account for this, what we will do is consider the time-rescaled random walk $X_{t/\mu}$ from now on. Next in line as mentioned in the heuristic section, is to explain how the fact that $\mathcal{H}$ is subcritical percolation gives us the desired upper bound on the displacement of $X_{t/\mu}$ on the time interval $[0,\beta]$. This just boils down to making the following observations:

1. If we let $C_{\mathcal{H}}(x)$ denote the connected cluster at x coming from the bond configuration $\mathcal{H}$, the exponential decay of the cluster size for subcritical percolation (which holds both in $\mathbf{Z}^d$ and $\mathbf{Z}_n^d$) gives us that for some $C > 0$,
$$\mathbf{P}\left[\text{diam}\left(C_{\mathcal{H}}(x)\right) \geq r\right] \lesssim \exp\left(-Cr\right).$$

2. The fast decay of the tail bound above immediately gives that for some $K > 0$,
$$\mathbf{E}\left[\text{diam}\left(C_{\mathcal{H}}(x)\right)^2\right] \leq K.$$

3. Finally, since the random walk X_t can only move through $C_{\mathcal{H}}(X_0)$ during the time interval $[0, \beta/\mu]$, it follows that dist$(X_0, X_{t/\mu}) \leq$ diam$(C_{\mathcal{H}}(X_0))$ for all $t \in [0,\beta]$. Putting this together, we have that for some constant $K > 0$:

$$\mathbf{E}\left[\text{dist}(X_0, X_{t/\mu})^2\right] \leq K \quad \text{for all } t \in [0,\beta] \tag{5.2}$$

Step 2: Markov-type of metric spaces

The next step on the itinerary is to extend 5.2 to all values of t. Let us introduce the following definition:

Definition 5.4 (Markov type 2)**.** *A metric space (S, d) is said to have Markov type 2 if there exists a constant C such that for every stationary reversible discrete-time Markov chain $(Z_n)_{n\geq 0}$ on a state space of finite size, and every function f from the state space to S, one has that for all $n \geq 0$:*

$$\mathbf{E}\left[d(f(Z_n), f(Z_0))^2\right] \leq Cn\mathbf{E}\left[d(f(Z_1), f(Z_0))^2\right].$$

As it turns out, the real line, satisfies this property, [13, Theorem 13.13]:

Proposition 5.5. *The real line has Markov type 2 with the constant C in Definition 5.4 being equal to 1.*

Proof. Let P be transition matrix of our Markov chain $(Z_n)_{n\geq 0}$ with invariant distribution π on state space Ξ with $|\Xi| = m$. Since the chain is assumed to be reversible, by the spectral decomposition Theorem, (e.g: see [11, Lemma 12.2]) we know there exists an orthornormal basis of eigenvectors $\{f_1, \cdots, f_m\}$ of $\mathbf{R}^{\Xi}$ and moreover, the eigenvalues will be contained in $[-1, 1]$. Then it is a standard computation, using stationarity, that

$$\begin{aligned}\mathbf{E}\left[(f(Z_t) - f(Z_0)^2\right] &= \sum_{i,j} \pi(i)P^t(i,j)\left(f(i) - f(j)\right)^2 \\ &= 2\left\langle (I - P^t)f, f\right\rangle_{\pi}\end{aligned}$$

so the claim reduces to showing that

$$\left\langle (I - P^t)f, f\right\rangle \leq t\left\langle (I - P)f, f\right\rangle$$

for all $f \in \mathbf{R}^{\Xi}$. If f is an eigenfunction with eigenvalue λ, then the claim reduces to simply showing that $(1-\lambda^t) \leq t(1-\lambda)$. By rearranging, and using the fact that $|\lambda| \leq 1$, this is in turn equivalent to showing that $1+\lambda+\cdots+\lambda^{t-1} \leq t$, which is obviously true. Otherwise, if f is not an eigenfunction, it can be decomposed as a sum of the orthonormal basis of eigenfunctions and the proof follows with a simple calculation. $\square$

Here we have for simplicity provided the proof of the case for the real line, but the same result also holds for the case of $\mathbf{R}^m$, with the Euclidean metric $\|\cdot\|$, [16, Lemma 4.2], i.e:

$$\mathbf{E}\left[\|f(Z_n) - f(Z_0)\|^2\right] \leq n\mathbf{E}\left[\|f(Z_1) - f(Z_0)\|^2\right] \tag{5.3}$$

Now we are ready to put all of the ingredients together.

1. First of all, we need a reversible, stationary, discrete-time Markov chain on a finite state space. If we impose that the system $M_t = (X_t, \eta_t)_{t\geq 0}$ is started at $u \otimes \pi_p$, then we can discretise time: let $t \geq \beta$, and choose $l \in \mathbf{N}$ be such that $v := \frac{t}{l} \in [\beta/2, \beta]$. Then we see that

$$Y_k := M_{\frac{kv}{\mu}}$$

   satisfies our conditions.

2. Now we seek to apply 5.3. For this we will need a suitable choice of $f : \mathbf{Z}_n^d \times \{0,1\}^{E(\mathbf{Z}_n^d)} \to \mathbf{R}^m$ for some m. As we are interested in talking about $\text{dist}(X_0, X_t)$, a natural choice is any function $f(x,\eta) = g_n(x)$, where $g_n : \mathbf{Z}_n^d \to \mathbf{R}^m$ is such that $\{g_n\}_{n\geq 0}$ are uniformly bi-Lipschitz in n when $\mathbf{Z}_n^d$ is equipped with the metric dist and $\mathbf{R}^m$ is equipped with the Euclidean metric. By uniformly bi-Lipschitz we mean that the function is bi-Lipschitz and the constant C_{Lip} is independent of n. An example of such a function is $g_n : \mathbf{Z}_n^d \to \mathbf{R}^{2d}$ given by

$$g_n(x_1, \cdots, x_d) = (n\cos(2\pi x_1/n), n\sin(2\pi x_1/n), \cdots, n\cos(2\pi x_d/n), n\sin(2\pi x_d/n)).$$

The actual form of the function is not too important, as long as it satisfies the uniform bi-Lipschitz condition above. Equipped with this function, we can now put everything together:

$$\begin{aligned}
\mathbf{E}\left[\text{dist}\left(X_0, X_{\frac{t}{\mu}}\right)^2\right] &\leq C_{\text{Lip}}^2 \mathbf{E}\left[\left\|g\left(X_{\frac{t}{\mu}}\right) - g(X_0)\right\|^2\right] &(5.4)\\
&= C_{\text{Lip}}^2 \mathbf{E}\left[\left\|f\left(Y_{\frac{lv}{\mu}}\right) - f(Y_0)\right\|^2\right] &(5.5)\\
&\leq C_{\text{Lip}}^2 l \mathbf{E}\left[\left\|f\left(Y_{\frac{v}{\mu}}\right) - f(Y_0)\right\|^2\right] &(5.6)\\
&\leq C_{\text{Lip}}^4 l \mathbf{E}\left[\text{dist}\left(X_0, X_{\frac{v}{\mu}}\right)^2\right] &(5.7)\\
&\leq K C_{\text{Lip}}^4 l &(5.8)\\
&\leq \frac{2KC_{\text{Lip}}^4}{\beta} t &(5.9)
\end{aligned}$$

Where 5.4 follows from the bi-Lipschitz condition, 5.5 is just plugging in the definition of v, 5.6 is the Markov type 2 property, 5.7 is once again the bi-Lipschitz condition (note how it was crucial that we have bi-Lipschitz instead of just Lipschitz), 5.8 follows from the fact that v/μ is in the "small time interval" $[0, \beta/\mu]$, and so we can apply 5.2, and finally, 5.9 follows from the fact that $l = t/v$ and $v \geq \beta/2$. And this finishes our proof, because since t was arbitrary (as long as it was large enough), choosing $t' = t\mu$, gives that for t large enough,

$$\mathbf{E}[\text{dist}(X_0, X_t)^2] \lesssim \mu t,$$

so in particular, we reach the final conclusion $\sigma^2 \lesssim \mu$ as desired. Note that β was independent of n and μ, and so was C_{Lip}, therefore the constant in 5.9 really does depend only on d and p. ☐

Remark 5.6 (Extending the result to $\mathbf{Z}^d$)**.** *Recall that this result has been proven for $G = \mathbf{Z}_n^d$, but this can easily now be extended to work in the whole of $\mathbf{Z}^d$, as shown in [16, Corollary 1.6]. We now give the heuristics of the argument: consider on the same probability space a random walk $(X_t)_{t\geq 0}$ on dynamical percolation on $\mathbf{Z}^d$, and for each n, a walk $(X_t^n)_{t\geq 0}$ that makes the same moves as (X_t), until it reaches the boundary of the box $[-n,n]^d$. At this point, it starts moving independently and sees its environment as if it were $\mathbf{Z}_n^d$. Then for a fixed time, we have that*

$$\textit{dist}(0, X_t) = \liminf_{n\to\infty} \textit{dist}(0, X_t^n),$$

and so by squaring and applying Fatou's Lemma we reach the desired conclusion.

An application to Mixing Times

Although the focus of this essay is mainly on studying the rates of diffusion of the random walk on dynamical percolation, a big part of the literature has been devoted to studying the mixing time of the system. We now present a tangential result to show how the result we have proven in Theorem 5.1 can be used to obtain a bound on mixing times. Recall that for an irreducible Markov chain $(Y_t)_{t\geq 0}$ on a finite state space Ξ with invariant measure π, with step distribution at time t $P^t(x,\cdot)$ when the chain is started from x, one can define its mixing time $t_{\mathsf{mix}}(\epsilon)$ as

$$t_{\mathsf{mix}}(\epsilon) = \inf\left\{t \geq 0 : \|P^t(x,\cdot) - \pi\|_{\mathsf{TV}} \leq \epsilon\right\},$$

where $\|\mu - \nu\|_{\mathsf{TV}} = \sup_{A\subset\Xi} \mu(A) - \nu(A)$ is the total variation distance between the two measures. Note however, that while we can therefore talk about the mixing time of the entire system (X_t, η_t), the same definition applied to $(X_t)_{t\geq 0}$ would not be the mixing time in the strict sense, as the process is not a Markov chain. Nonetheless, there is still nothing that stops us from obtaining a bound on a "mixing time" for $(X_t)_{t\geq 0}$. This is the content of the following proposition [16, Theorem 1.2 (ii)]:

Proposition 5.7. *For $p < p_c$ and any $d \geq 1$ and $\epsilon > 0$, there exists a constant $C > 0$ and an integer $N > 0$ such that whenever $n \geq N$, and for all μ,*

$$\inf\left\{t \geq 0 : \|\mathcal{L}(X_t) - u\|_{TV} < \epsilon\right\} \geq \frac{Cn^2}{\mu}$$

Where the initial bond configuration is π_p and $\mathcal{L}(X_t)$ is the law of the random walk on dynamical percolation on $\mathbf{Z}_n^d$ at time t started at zero, and u is the uniform distribution on $\mathbf{Z}_n^d$.

We now explain the heuristics of this proof:

Important Idea ☕. *Since we have an upper bound on how much the particle is expected to move in a given period of time, for small enough values of time, we can find a ball of radius small enough so that the particle will be with high probability concentrated in a set of low probability. This will mean that the particle can't be mixed.*

Proof. It suffices to show that there exists constants $C = C(d,p,\epsilon)$ and $N = N(d,p,\epsilon)$ such that whenever $s < \frac{Cn^2}{\mu}$, one has that $\|\mathcal{L}(X_s) - u\|_{\mathsf{TV}} \geq \epsilon$. We will do this by noting that the definition of total variation distance immediately implies that for any set $E_n \subset \mathbf{Z}_n^d$ of our choice, we will have that $\|\mathcal{L}(X_s) - u\|_{\mathsf{TV}}$ is lower bounded by $\mathbf{P}_{\delta_0\times\pi_p}[X_t \in E_n] - u(E_n)$. The goal is therefore to find a set of small probability such that the particle is likely to be found in E_n by time s. In light of the mean square displacement bounds we have obtained, a natural choice is to set

$$E_n = \left\{x \in \mathbf{Z}_n^d : \mathsf{dist}(0,x) \leq bn\right\},$$

where b is a constant to be determined. As we have remarked before, the symmetry of the torus implies that the mean square displacement bounds also hold for when the starting distribution is not uniform, so in fact:

$$\mathbf{P}_{\delta_0\times\pi_p}[X_s \in E_n] = 1 - \mathbf{P}_{\delta_0\times\pi_p}[\text{dist}(X_0, X_s) > bn] \tag{5.10}$$

$$\geq 1 - \frac{\mathbf{E}\left[\text{dist}(X_0, X_s)^2\right]}{(bn)^2} \tag{5.11}$$

$$\geq 1 - \frac{K(Cn^2 \vee 1)}{(bn)^2} \tag{5.12}$$

Where K is the constant coming from the upper bound on mean square displacement. Moreover, the size of E_n is bounded above by $(2bn)^d$ and so $u(E_n)$ is at most $(2b)^d$. Therefore, if we choose n and b so that

$$\frac{K}{(bn)^2} < \frac{1-\epsilon}{2} \quad \text{and } (2b)^d < \frac{1-\epsilon}{2},$$

we can combine it with 5.12, and obtain that in fact

$$\|\mathcal{L}(X_s) - u\| \geq \frac{1+\epsilon}{2} - \frac{1-\epsilon}{2} = \epsilon$$

as required. □

Remark 5.8. *As some concluding remarks on the discussion of mixing times, we should point out that in the original paper by Peres, Stauffer, and Steif, an upper bound for the mixing time of the whole system was found, and it is up to constants also n^2/μ. The lower bound we have just shown therefore implies that the mixing time of the entire system $(X_t, \eta_t)_{t\geq 0}$ is of order $n^2\mu$. The proof of this upper bound - substantially more technical than the lower bound - uses a coupling argument and relies on the use of regeneration times.*

Small detour: a general upper bound

In the proof of Proposition 4.1, where we established the connection between σ^2 and the mean square displacement of the particle, there was a step in which we employed a general upper bound on $\mathbf{E}[\text{dist}(0, X_t)^2]$ (technically we bounded $\mathbf{E}[d(0, X_t)^2]$ but as mentioned before, these two quantities are the same up to constants since the ℓ^1 distance and the Euclidean distance are equivalent metrics). We delayed the proof of this fact until now since it requires the use of a Markov-type argument, which for pedagogical reasons was best introduced in the proof of Theorem 5.1. We now make this bound precise:

Proposition 5.9 (A general upper bound)**.** *Let $(X_t)_t$ be a random walk on dynamical percolation on $\mathbf{Z}^d$ with any parameters $\mu > 0$ and $p \in [0, 1]$. Then*

$$\mathbf{E}[\textit{dist}(0, X_t)^2] \lesssim t.$$

Proof. The proof follows almost immediately now that we have the Markov-type property in our toolbox. The only thing to notice is that since $(X_t)_t$ jumps at rate 1, dist(X_0, X_t) is stochastically dominated by a Poisson random variable with parameter t, from which it follows that whenever $t \in [0, 1]$

$$\mathbf{E}[\text{dist}(X_0, X_t)^2] \leq t + t^2 \leq 2.$$

This is essentially the same setup we had in Equation 5.2. Now one can run the exact same argument we did in the proof of Theorem 5.1 in order to extend to all values of time the bound on the mean square displacement obtained from the exponential decay of the cluster size. □

6 Critical Regime

We now reach the next stop in our study of the behavior of σ^2 and v. Before saying anything else, let us state some results: for the case of Euclidean lattices, we will be able to provide bounds for σ^2 on the triangular lattice $\mathcal{T}$, as well as on $\mathbf{Z}^d$, but only for $d = 2$, or $d \geq 11$. As we will explain shortly, the proofs for each of these cases are essentially the same.

Theorem 6.1 (Diffusion constants at criticality)**.** *Let $p > p_c$ and consider a random walk on dynamical percolation on a graph G with initial bond configuration distributed according to π_p. Then:*

1. *When $G = \mathcal{T}$, we have that*
$$\sigma^2 \lesssim \mu^{\frac{5}{132}+o(1)}$$
2. *When $G = \mathbf{Z}^d$:*
   (a) *If $d = 2$, there exists some $\delta \in (0,1)$ such that*
   $$\sigma^2 \lesssim \mu^{\delta}.$$
   (b) *If $d \geq 11$, then*
   $$\sigma^2 \lesssim \mu^{1/2}\log(1/\mu)$$

This was one of the main results of [5]. We will state the corresponding result for the speed after we give the heuristics of the proof of this Theorem.

Structure of the proof and heuristics

In the subcritical case, we saw that our biggest ally was the fact that for small times, the walker gets trapped inside a cluster whose size we can control very well, and then one could extend the result to all times by using a Markov-type argument. Of course, now that we are working at criticality we can't replicate this method, but since ideas in mathematics are expensive, we will try to adapt known facts from critical percolation so that we can employ this tactic here too. Let us explain what the main tool will be:

Important Idea ☕. *A one-arm estimate for percolation at parameter p is a pair of bounds that control from above and below the probability that the origin is connected to the boundary of some large box. For $p = p_c$, we have a collection of such bounds for different kinds of graphs, but with some work, one can modify the bounds so that they give us control over the case where percolation is not only critical, but "slightly supercritical". In precise terms, we will have constants a_1, a_2, C_1, C_2, and v, all in $(0,\infty)$ such that*
$$\frac{C_1}{r^{a_1}} \leq \mathbf{P}_p\left[0 \longleftrightarrow \partial\Lambda_r\right] \leq \frac{C_2}{r^{a_2}} \qquad \text{for any } p \in \left[p_c(G), p_c(G) + r^{-\frac{1}{v}}\right] \tag{6.1}$$

Provided that an estimate of this form holds, we will proceed just as we did for the proof of Theorem 5.1: one can look at the bond configuration formed by the edges that have been open at least once in dynamical percolation during some small time interval $[0,t]$, and calling this set of open edges $\mathcal{H}$, we shall see that $\mathcal{H}$ will come from supercritical percolation, but only *slightly supercritical* if we choose t to be small enough. We will then be in good shape to apply our one-arm-estimate assumption and obtain the bounds we seek. The way this is done is as follows:

Important Idea ☕. *To bound the motion of the particle, we can distinguish different cases depending on how big $\mathcal{H}$ is:*

- *If $\mathcal{H}$ is small, we can bound the distance travelled by the particle simply by using the diameter of the cluster, and then apply the one-arm-estimate to bound the probability of having a cluster of said diameter.*
- *If $\mathcal{H}$ is big, we can apply a general upper bound on the displacement of the particle, perhaps not very sharp a priori, but improved when applying the one-arm-estimate to account for the small probability of having a large diameter.*

Once we have the desired bounds on this small time interval $[0,t]$, then one can extend this to all values of time

In the case of the Euclidean lattices, we will be able to explicitly write down a one-arm-estimate of the form 6.1, whereas for the speed, since we work on more general graphs, we also have an upper bound, but when working under the assumption that an analogue to 6.1 holds:

Theorem 6.2 (Upper bound of speed at criticality)**.** *Let G be a connected, locally finite, transitive graph where each vertex has degree $d \geq 3$. Suppose it satisfies a one arm estimate: for all $v \in G$,*

$$\mathbf{P}_{p_c}[\mathit{Rad}_{\mathit{int}(C_v)} \geq r] \leq \frac{C}{r},$$

Where C_v is the open cluster containing v, and $\mathit{Rad}_{\mathit{int}(C_v)}$ is the intrinsic radius (i.e: maximum over $w \in C_v$ of the length of self avoiding open paths connecting w and v). Let the initial bond configuration be distributed according to π_p, then we have for all $\mu \in (0, 1/e]$:

$$v_{p_c}(\mu) \lesssim \sqrt{\mu \log(1/\mu)}.$$

As we have just explained, the proofs of our two main results - the upper bound on σ^2 for the Euclidean lattices, and the upper bound for the speed on more general graphs - both have the following three-step structure:

1. Extend a one-arm-estimate known (in the case of the Euclidean lattices), or assumed (in the case of more general graphs), that works at $p = p_c$ so that it also holds in "slight-supercriticality".
2. Use this extension to control the motion of the walk: either the square displacement (for the bound on σ^2) or the linear displacement (for the bound on v) on some fixed time interval.
3. Extend the bounds to all values of time.

Since the proof strategy is shared between both σ^2 and v, we will present each step in the setting where it is more pedagogical:

- Step 1 will be shown in the setting of Theorem 6.1, and in particular, as inspired by Peres' talk at ICBS 2024, we will explain the proof in the high-dimensional case due to its cleaner appearance.
- Step 2 will be shown in the context of Theorem 6.2, where the displacement estimates require simpler computations but still convey the main ideas.
- Step 3 is quite simple in comparison to the previous ones, so we will quickly explain it for both cases.

Step 1: The Kozma-Nachmias esimate and its extension

As we have just explained, we will now explain Step 1 by illustrating the example of high dimensions in $\mathbf{Z}^d$. The end-goal of this step is to obtain an estimate of the form 6.1, which gives us control on the probability that the origin is connected to the boundary of the box of radius r for values of p that are allowed to be slightly above p_c. The story begins with the following Theorem: [9, Theorem 1].

Theorem 6.3 (Kozma-Nachmias)**.** *Let $d \geq 19$, then there are constants a and b in $(0, \infty)$ such that*

$$\frac{a}{r^2} \leq \mathbf{P}_{p_c}\left[0 \longleftrightarrow \partial\Lambda_r\right] \leq \frac{b}{r^2}.$$

Where $\mathbf{P}_{p_c}$ is the bond percolation measure at p_c.

The particularly clean form of the Kozma-Nachmias estimate is why we are illustrating Step 1 in the setting of high dimensions. We will also need the following Theorem [4, Theorem 1.6]:

Theorem 6.4 (van der Hofstad-Fitzner)**.** *Let $d_p(x, y)$ denote the length of the shortest open path between x and y in bond percolation in $\mathbf{Z}^d$ with parameter p and $d \geq 11$. Then there is a constant c for which*

$$\mathbf{P}\left[\text{there is an } x \in \mathbf{Z}^d \text{ with } d_{p_c}(0, x) = r\right] \leq \frac{c}{r}$$

for all $r \in \mathbf{N}$, and where $\mathbf{P}$ is the probability measure on the space where the percolation model is defined.

The rest of the work comes in now to show that one can extend this result to obtain a version of the estimate 6.1. We follow the proof strategy from [5, Lemma 2.9]. Our goal will now be to prove that in fact, there are constants C_1 and C_2 such that

$$\frac{C_1}{r^2} \leq \mathbf{P}_p\left[0 \longleftrightarrow \partial\Lambda_r\right] \leq \frac{C_2}{r^2} \quad \text{for all } p \in \left[p_c, p_c + r^{-1/2}\right]. \tag{6.2}$$

Let p be in the range specified in 6.2. The lower bound follows immediately from Theorem 6.3 since the event $\{0 \longleftrightarrow \partial\Lambda_r\}$ is increasing and $p \geq p_c$. The upper bound will require more thought. We can first do the following trivial bound:

$$\begin{aligned} \mathbf{P}_p\left[0 \longleftrightarrow \partial\Lambda_r\right] \leq\ & \mathbf{P}\left[\text{there is an } x \in \partial\Lambda_r \text{ with } d_p(0, x) \leq r^2\right] \\ & + \mathbf{P}\left[\text{there is an } x \in \partial\Lambda_r \text{ with } r^2 < d_p(0, x) < \infty\right]. \end{aligned} \tag{6.3}$$

Let us now take care of the first term in equation 6.3. The key idea is the following: we are currently dealing with supercritical percolation, and we wish to relate it to critical percolation so that we can use Theorems 6.3 and 6.4. Fortunately, there is a way to obtain p_c-percolation when starting from p-percolation. Simply sample p-percolation and then close each open edge independently from each other with probability $1 - \frac{p_c}{p}$. Formally, we can define a p-percolation model η in some probability space and then we construct our p_c-percolation $\widehat{\eta}$. by selecting a family $\{U_e\}_{e \in E(\mathbf{Z}^d)}$ of i.i.d uniform random variables on the same probability space, and defining

$$\widehat{\eta}(e) = \mathbf{1}\left(\eta(e) = 1, U_e \leq \frac{p_c}{p}\right).$$

Then it is clear that if η is a bond configuration for which there is an $x \in \partial\Lambda_r$ such that the length of the shortest open path Γ between 0 and x is at most r^2, and moreover, for each edge $e \in \Gamma$, one has that

$U_e \leq \frac{p_c}{p}$, then we have that $\widehat{\eta}$ is also a bond configuration for which there is some $x \in \partial\Lambda_r$ such that the length of the shortest open path between 0 and x is at most r^2. Putting all of this together, and since η and the family $\{U_e\}_{e\in E(\mathbf{Z}^d)}$ are independent, the length of Γ is at most r^2, and $\widehat{\eta}$ is p_c-percolation, it follows that

$$\mathbf{P}\left[\text{there is an } x \in \partial\Lambda_r \text{ with } d_p(0,x) \leq r^2\right]\left(\frac{p_c}{p}\right)^{r^2} \leq \mathbf{P}\left[\text{there is an } x \in \partial\Lambda_r \text{ with } d_{p_c}(0,x) \leq r^2\right],$$

now using the fact that $p \leq p_c + r^{-2}$, and the Theorem of Kozma and Nachmias 6.3, we see that,

$$\mathbf{P}\left[\text{there is an } x \in \partial\Lambda_r \text{ with } d_p(0,x) \leq r^2\right] \leq \left(\frac{p_c + r^{-2}}{p_c}\right)^{r^2} \times \frac{b}{r^2} \tag{6.4}$$

Noting that the fraction in 6.4 with the r^2 in the exponent is indeed bounded by a constant that depends only on p_c, we get the advertised bound for the first term of the sum in 6.3. The bound on the second term in the sum of 6.3 follows with the same ideas. Namely:

$$\begin{aligned}\mathbf{P}\left[\text{there is an } x \in \partial\Lambda_r \text{ with } r^2 < d_p(0,x) < \infty\right] &\leq \mathbf{P}\left[\text{there is a } y \in \Lambda_r \text{ with } d_p(0,y) = r^2\right] \\ &\leq \left(\frac{p_c + r^{-2}}{p_c}\right)^{r^2} \times \frac{c}{r^2}\end{aligned} \tag{6.5}$$

Where we used in the second inequality a similar argument as before to translate from p-percolation to p_c-percolation, as well as Theorem 6.4. Finally, combining 6.3 and 6.5 into 6.4 finishes the claim. □

Remark 6.5 (The graphs $\mathcal{T}$ and $\mathbf{Z}^2$:). *The only difference between the exposition we have given here and the corresponding estimates for the case of $\mathcal{T}$ and $\mathbf{Z}^2$ is that in each of these cases we will have different analogues of Theorem 6.3 at our disposal. The argument to extend these bounds to slight-supercriticality is essentially the same. These different one-arm-estimates at criticality are exactly what give rise to the slightly different upper bounds in each case in Theorem 6.1. For completeness, let us state the extended one-arm-estimates for the remaining two cases: for site percolation on the triangular lattice $\mathcal{T}$, one has that for all $\epsilon \in (0, 5/48)$, there exists constants C_0 and C_1 both in $(0,\infty)$ such that for $r \geq 1$,*

$$C_0 r^{-\frac{5}{48}-\epsilon} \leq \mathbf{P}_p\left[0 \longleftrightarrow \partial\Lambda_r\right] \leq C_1 r^{-\frac{5}{48}+\epsilon} \quad \text{for all } p \in \left[\frac{1}{2}, \frac{1}{2} + r^{-\frac{(3+\epsilon)}{4}}\right].$$

On the other hand, for bond percolation on $\mathbf{Z}^2$ there are constants $a, C_0, C_1 \in (0,\infty)$ such that for $b = -\frac{1}{2}\log\left(1 - 2^{-24}(1-\sqrt{3}2)^{48}\right)$, one has for $r \geq 1$

$$\frac{C_0}{r^a} \leq \mathbf{P}_p[0 \longleftrightarrow \partial\Lambda_r] \leq \frac{C_1}{r^b} \quad \text{for all } p \in \left[\frac{1}{2}, \frac{1}{2} + r^{-2}\right].$$

Remark 6.6 (One-arm-estimate in Theorem 6.2). *Here we have presented this extension argument for the case of Euclidean lattices. However, these same ideas can also be carried out in a similar fashion to extend the one-arm-estimate assumed to hold in Theorem 6.2 into slight supercriticality. In particular, as shown in [6, Lemma 3.1] we have that under the hypothesis of Theorem 6.2, there are constants C and δ such that*

$$\mathbf{P}_p[\mathit{Rad}_{\mathit{int}}(\mathcal{C}_v) \geq r] \leq \frac{C}{r} \qquad \text{for } p \in [p_c, p_c + \delta/r].$$

Step 2: Control of $\mathbf{E}[\mathsf{dist}(0, X_t)]$ in a small time interval

Our next stop is to use our modified one-arm-estimate to start bounding the displacement of the particle on a small time window. As explained before, we will now switch gears and present this argument in the case of the speed (Theorem 6.2), so that G is a transitive graph with bounded degree. We will fix some vertex $o \in V(G)$ to be the root, and assume that the random walk starts at o. The reason why we present this step in the setting of the speed instead of the diffusion constant is merely that the computations we'll have to do take a cleaner form. The ideas extend almost exactly the same to the other case, with the exception of a general upper bound which we will remark later. We start by making the following comments:

- Using the same notation as in Section 5, let $\mathcal{H}$ be the subgraph of G induced by the bonds that have opened at least once during the time interval $[0, t]$, where t will be chosen in the end. Since the initial environment is distributed according to π_p, doing the same calculation as in equation 5.1, we have that an edge $e \in E(G)$ belongs to $\mathcal{H}$ with probability at most $p_c(1 + \mu t)$.

- Let $\mathbf{P}_p$ be percolation measure with parameter p on the graph G, and for a vertex $v \in V(G)$, let $\mathcal{C}_v$ be the open cluster of v. Recall that in view of Remark 6.6, we have constants C and δ such that

$$\mathbf{P}_p[\mathsf{Rad}_{\mathsf{int}}(\mathcal{C}_v) \geq r] \leq \frac{C}{r} \qquad \text{for } p \in [p_c, p_c + \delta/r]\,.$$

  In particular, letting $\mathcal{H}_o$ be the open cluster of the root of the graph, we have that if $p_c \mu t \leq \delta/r$, then

$$\mathbf{P}[\mathsf{Rad}_{\mathsf{ext}}(\mathcal{H}_o) \geq r] \leq \frac{C}{r}, \tag{6.6}$$

  where $\mathsf{Rad}_{\mathsf{ext}}(\mathcal{C}_v)$ is the maximum over $w \in \mathcal{C}_v$ of the graph distance between v and w, and $\mathbf{P}$ is now the probability measure of the space on which the random walk on dynamical percolation is defined.

- With 6.6 in mind, we are ready to control the displacement of the particle in a small time interval. The idea is to divide the motion of the particle into different cases depending on the extrinsic radius of $\mathcal{H}_o$. On the events that $\mathsf{Rad}_{\mathsf{ext}}(\mathcal{H}_o)$ is small, we can obtain a good bound on the displacement of the particle using the facts the particle is trapped inside a small cluster. On the other hand, on the event that $\mathcal{H}_o$ is quite big, we can use a general upper bound for $\mathbf{E}[\mathsf{dist}(0, X_t)]$, which although not very sharp, gives us good control when combined with the fact that there is a low probability that $\mathcal{H}_o$ is big. Let us now make this precise: let K be some threshold which we will determine in a moment, we can divide the motion of X into the following cases:

$$\begin{aligned} \mathbf{E}[\mathsf{dist}(X_0, X_t)] \leq & \sum_{k=1}^{K} \mathbf{E}\left[\mathsf{dist}(X_0, X_t)\mathbf{1}\left\{2^{k-1} \leq \mathsf{Rad}_{\mathsf{ext}}(\mathcal{H}_o) \leq 2^k\right\}\right] \\ & + \mathbf{E}\left[\mathsf{dist}(X_0, X_t)\mathbf{1}\left\{\mathsf{Rad}_{\mathsf{ext}}(\mathcal{H}_o) \geq 2^k\right\}\right] \end{aligned} \tag{6.7}$$

  if K is chosen such that

$$\frac{\delta}{2^{K+1}} < p_c \mu t \leq \frac{\delta}{2^K},$$

  then we'll have that $p_c \mu t < \frac{\delta}{2^k}$ for all $k = 1, \cdots, K$, and so we can use 6.6 to conclude that the first term in 6.7 is bounded above by

$$\sum_{k=1}^{K} 2^k \times \frac{C}{2^{k-1}} = 2CK.$$

  Now we take a look at the remaining term of 6.7. On event, we will use the fact that the distance is dominated by a Poisson process of rate 1, to obtain the bound $\mathbf{E}[\mathsf{dist}(0, X_t) \mid \mathcal{H}_o] \leq t$ and deduce that the second term of 6.7 is bounded above by $\frac{Ct}{2^K}$.

- Combining everything gives that in fact

$$\begin{aligned}\frac{1}{t}\mathbf{E}[\mathsf{dist}(X_0,X_t)] &\leq \frac{2CK}{t}+\frac{C}{2^K}\\ &\leq 2C\left(\frac{-\log_2(p_c\mu t/\delta)}{t}+\frac{p_c\mu t}{\delta}\right),\end{aligned}$$

where for the second inequality we used the lower bound we have from our choice of K. We now optimise this by choosing $t=t(\mu)=\sqrt{(1/\mu)\log(1/\mu)}$, and obtain that for this choice of t:

$$\frac{1}{t}\mathbf{E}[\mathsf{dist}(X_0,X_t)] \leq C\sqrt{\mu\log(1/\mu)} \tag{6.8}$$

Remark 6.7 (Comparison of this step with the case of Theorem 6.1)**.** *We have shown how to control $\mathbf{E}[\mathit{dist}(0,X_t)]$ for a concrete value of t, but before we extend this result to all values of t, let us say some words about how this proof compares with the corresponding proof of Step 2 for Theorem 6.1. The ideas are essentially the same: consider the motion of the particle depending on the size of the cluster formed by bonds that have opened at least once during some small time interval, i.e:*

$$\begin{aligned}\mathbf{E}[\mathit{dist}(0,X_t)^2] &= \sum_{k=1}^{K}\mathbf{E}\left[\mathit{dist}(0,X_t)^2\mathbf{1}\left\{0\leftrightarrow\partial\Lambda_{2^{k-1}},0\not\leftrightarrow\partial\Lambda_{2^K}\right\}\right]\\ &\quad+\mathbf{E}\left[\mathit{dist}(0,X_t)^2\mathbf{1}\left\{0\leftrightarrow\partial\Lambda_{2^K}\right\}\right]\end{aligned}$$

the first part of this expression represents the case where the clusters are small, and just like we did here, we can bound the mean square displacement using the size of the cluster. However, the remaining term, corresponding to the rare event where the cluster is quite large, needs some more careful analysis. The idea is, just like we did here, to obtain a general bound. In this case, what one obtains is something of the form

$$\mathbf{E}\left[\mathit{dist}(0,X_t)^2\mathbf{1}\left\{0\leftrightarrow\partial\Lambda_{2^K}\right\}\right]\leq(t\log t)\mathbf{P}[0\leftrightarrow\partial\Lambda_{2^K}].$$

The proof of this upper bound can be found in [5, Propositions 2.2,2.4]. Once this upper bound is established, one chooses K and t in a suitable way so that one obtains the desired bounds in some time interval $[0,t]$.

Step 3: Extending to all values of t

Recall that in the previous step we saw that for $t(\mu)=\sqrt{(1/\mu)\log(1/\mu)}$, we obtained bound 6.8. Extending this result to obtain the corresponding bound on $v_{p_c}(\mu)$ is not difficult, we simply perform the following computation:

$$\begin{aligned}v_{p_c}(\mu) &= \lim_{T\to\infty}\frac{\mathbf{E}[\mathsf{dist}(X_0,X_T)]}{T}\\ &\leq \lim_{T\to\infty}\frac{t}{T}\sum_{n=0}^{\lfloor T/t\rfloor}\frac{1}{t}\mathbf{E}[\mathsf{dist}(X_{nt},X_{(n+1)t})]\leq C\sqrt{\mu\log(1/\mu)}\end{aligned}$$

where the first inequality is just the triangle inequality and for the second inequality we used the fact that the environment always has the stationary distribution, so that the expected displacement of the particle between times nt and $(n+1)t$ is in effect the same expected displacement between times 0 and t.

Remark 6.8 (Extending the bound in Theorem 6.1)**.** *As mentioned in Step 2, in the proof of Theorem 6.1, one obtains an interval of time $[0,t]$ for which the advertised bounds hold. At this point, one performs a Markov-type argument, exactly in the same fashion as we did in Section 5 to finish the claim. We will not go into details as the computations are essentially the same as the ones we saw in the subcritical case.*

7 Supercritical Regime: informal comments

We conclude this essay with a brief discussion of the behavior of σ^2 and v in the case where $p > p_c$. Since the methods of proof are somewhat different to the ones used in the previous two cases, making an exposition equally as detailed as the ones presented earlier would require considerably more space than that available, so we will limit ourselves to an informal overview of the main results.

Intuitively, in the subcritical case the particle was trapped inside clusters of finite size, and had to wait a time of order $1/\mu$ for the cluster to undergo noticeable changes and thus be able to explore other areas of the graph. This is why one could expect, as we saw in Section 5, that the speed and diffusivity constants were order μ. In the supercritical case however, there will always be a positive probability that the particle is in an infinite cluster, so that in some sense, the behavior of the particle wouldn't be too influenced by the evolution of the graph. Hence we may expect these quantities to be of order 1. The following is the result concerning σ^2: [5, Theorem 1.5]

Theorem 7.1 (Diffusion constant in supercritical regime). *Fix d and let $p \in (p_c, 1]$. Then for dynamical percolation on $\mathbf{Z}^d$ started at zero and initial bond distribution π_p, there is a constant $c > 0$ independent of μ, such that*

$$c \le \sigma^2 \le 1$$

With the corresponding result [6, Theorem 1.2] for v:

Theorem 7.2. *Let G be a connected, locally finite, transitive, non-amenable, and unimodular graph, where each vertex has degree at least 3. For any $p > p_c(G)$, there exists a constant c independent of μ, such that*

$$c \le v \le 1.$$

Remark 7.3 (Upper bounds). *Both upper bounds are immediate from what we know already: in the case of σ^2 in $\mathbf{Z}^d$, the upper bound is just the fact that $\mathbf{E}[dist(0, X_t)^2] \lesssim t$, and the upper bound for v follows immediately from the fact that dist$(0, X_t)$ is stochastically dominated by a Poisson random variable of mean 1, and from Theorem 4.3.*

Remark 7.4 (Lower bounds). *The main tool used in the lower bounds is the evolving set process: given a Markov Chain X with transition matrix P on a state space V and invariant distribution π, the evolving set process is a Markov chain on 2^V with the following transition rules: suppose that the current state of the process is $S_n = A$, then S_{n+1} is defined by*

$$S_{n+1} = \left\{ y \in V : \sum_{x \in A} \pi(x) P(x, y) \ge U_{n+1} \pi(y) \right\},$$

where $(U_n)_{n \ge 0}$ be a sequence of i.i.d uniform random variable on $[0, 1]$. The power of this method comes from the fact that one can couple both chains in such a way that sampling uniformly from the evolving set gives the same distribution as that of the random walk. The key fact now is that in supercritical percolation, one can use isoperimetric arguments to show that the evolving set grows very quickly, this gives the lower bounds. Detailed expositions of this procedure can be found in [5, Section 4], and [6, Section 3] for σ^2 and v respectively.

Remark 7.5 (Separating critical and supercritical case). *Notice that the fact that these lower bounds are independent of μ, in contrast to the upper bounds of Section 6, gives us a separation between the critical regime and the supercritical regime as we take $\mu \to 0$.*

8 Conclusion

In this essay we have studied random walks on dynamical percolation. The essay started by providing a general introduction to the model, and some of its main properties that showcased similarities with usual random walks on $\mathbf{Z}^d$. This initial discussion culminated with the proof of a Central Limit Theorem for random walks on dynamical percolation, leading to the fact that despite the underlying randomness of the environment, this model exhibits diffusivity.

This motivated us to study the question of how the diffusivity and speed change as the percolation parameter p ranges through the three regimes. By focusing on the key results of recent works such as [5], and [6], as well as more foundational papers such as [16], we provided an overview of all results known to date regarding the diffusion constant and the speed, showcasing the key ideas behind the proofs of some of the main results.

To conclude, we mention two open questions, as noted in Peres' ICBS 2024 talk: although we have upper and lower bounds for the critical and supercritical case respectively that separate the behavior of the two regimes, we still don't know what the sharp exponents for the upper bound in the critical case should be. Additionally, while the bounds presented in all regimes were all monotone in μ and p, it is now known whether the maps $(\mu, p) \mapsto \sigma^2(\mu, p)$ and $(\mu, p) \mapsto v_p(\mu)$ are themselves also monotone increasing.

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